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Ground state properties and excitation spectrum of the degenerate supersymmetric t - J model in one dimension

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Abstract. We consider the one-dimensional $SU(N)$ -invariant t - J model, which consists of electrons with N spin components on a lattice with nearest-neighbour hopping t constrained by the excluded multiple occupancy of the lattice sites and spin-exchange J between neighbouring lattice sites. The model is integrable at the supersymmetric point $t = J$. The ground state Bethe *ansatz* equations are analysed and solved numerically for arbitrary band filling and several values of N . The ground state energy, the chemical potential and the spin susceptibility are obtained as a function of band filling. The elemental charge and spin excitations are derived for arbitrary N and band filling. The energy of the charge excitations vanishes at the Fermi surface. The Fermi velocity has a maximum as a function of band filling, vanishing for the empty and full bands. The spinwave velocity is inversely proportional to the susceptibility. For exactly one electron per site the charge fluctuations are suppressed and the Bethe *ansatz* equations map onto those of the $SU(N)$ -invariant Heisenberg chain.

1. Introduction

The two-dimensional Hubbard model is believed to possess the main features necessary to explain many of the fundamental low-energy properties of the cuprate high-temperature superconductors [1]. The key ingredient is the motion of highly correlated electrons within the CuO planes. This refers only to low-energy properties, since the higher-energy excitations, as measured e.g. by photoemission, involve several bands and their complicated dispersions. In the limit of very large on-site Coulomb repulsion the Hubbard model can be mapped onto the t - J model with $t \gg J$, for which numerous properties have been studied with approximate methods [2, 3]. It has been conjectured [1, 4] that the 1D and 2D variants of both the Hubbard and t - J models, have properties in common. Exact results in 1D are often more accessible than 2D ones and may provide a testing ground for approaches intended for more complex problems. In this paper we solve exactly the degenerate supersymmetric 1D t - J model, for which there is no direct application to experiments.

The one-dimensional t - J model for spin $\frac{1}{2}$ was found to be integrable at the supersymmetric point by Sutherland [5]. This supersymmetry is related to the $SU(3)$ -invariant Heisenberg chain of spin 1. The Bethe *ansatz* equations, the classification of states and the thermodynamic equations for this model were presented in [6]. These results were used by Bares and Blatter [7, 8] to calculate explicitly the spectrum of elementary excitations and by Kawakami and Yang [9] to obtain the exponents

for the long-distance asymptotic of correlation functions. In [10] we constructed an asymptotically exact solution in the low-electron-density limit for coupling parameters deviating from $SU(3)$ symmetry, i.e. $t \neq J$ and Lee and Schlottmann [11] extended the Bethe *ansatz* solution at the supersymmetric point to an arbitrary number of spin components N . The thermodynamic Bethe *ansatz* equations for the N -fold degenerate supersymmetric t - J model were derived and discussed by Schlottmann [12]. The spin and charge excitations of the one-dimensional degenerate Hubbard model as well as the properties of the associated metal-insulator transition at non-zero U (for $N > 2$) were presented in [13, 14].

In this paper we extensively discuss the ground state Bethe *ansatz* equations and derive the excitation spectrum of the 1D supersymmetric t - J model with N spin components ($N = 2S + 1$), generalizing in this way the results of [7, 8, 12]. The procedure to be followed is in complete analogy with that described in [13].

The model under consideration is the following

$$H = - \sum_{is} P (c_{is}^\dagger c_{i+1s} + c_{i+1s}^\dagger c_{is}) P + V \sum_{iss'} n_{is} n_{i+1s'} + J \sum_{iss'} c_{is}^\dagger c_{is'} c_{i+1s'}^\dagger c_{i+1s} \quad (1.1)$$

where the hopping matrix element t has been equated to 1. Here c_{is} annihilates an electron at the site i with spin component s , ($|s| \leq S$), n_{is} is the number operator, P is a projector excluding the *multiple occupancy* of every site, V is a charge interaction independent of the spin and J is a spin exchange interaction. The generalized spin S can be thought of as composed of spin and orbital degrees of freedom.

Several special cases of this model are worth noting.

(i) If $J = 0$ the spin components do not play a relevant role, and we only have to distinguish occupied and empty sites. The model then reduces to the spin- $\frac{1}{2}$ anisotropic XXZ -Heisenberg chain.

(ii) If there is one electron per site there is no dynamics of charges and the system is just the Heisenberg chain of spin S with $SU(2S + 1)$ invariance [5]. In addition, model (1.1) is integrable for arbitrary band filling in the following cases:

(iii) $J = -V = \pm 1$

(iv) $J = V = \pm 1$.

Situations (iii) and (iv) are related to $SU(2S + 2)$ invariance of the model with $(2S + 1)$ spin and one charge degree of freedom. The situation $J = -V = 1$ corresponds to the supersymmetric limit of the traditional t - J model [11] (the t - J model represents the high-correlation limit of the degenerate Hubbard model only if $J \ll t$). The ground state properties and the excitation spectrum of this case for arbitrary spin are the subject of this paper.

The rest of the paper is organized as follows. In section 2 we restate the discrete Bethe *ansatz* equations derived previously in [11]. The structure of the ground and excited states is discussed and integral equations relating the densities of the corresponding rapidity sets [12] are restated for the zero-temperature limit. The zero-field ground state properties as obtained by numerically solving the ground state integral equation for the charges are discussed in section 3. The elemental charge (adding or removing one particle from the system) and spin excitations are derived in sections 4 and 5, respectively, for arbitrary band filling and degeneracy N . A summary and conclusions are presented in section 6.

2. Bethe ansatz equations

The diagonalization of the Hamiltonian (1.1) for $J = -V = 1$ is a straightforward generalization of the procedure presented in [6]. The two-particle scattering matrix is given by

$$S(k_1, k_2) = \frac{p_1 - p_2}{p_1 - p_2 + i} \mathbf{1} + \frac{i}{p_1 - p_2 + i} \mathbf{P} \tag{2.1}$$

where $\mathbf{1}$ is the identity matrix and \mathbf{P} permutes the spin indices of the two electrons. Here p is related to the wavenumber k by $p = \frac{1}{2} \cot(k/2)$. It is easy to verify that (2.1) satisfies the triangular Yang-Baxter relation [15] and that the scattering matrix for a multiparticle scattering process can be written as a product of two-particle scattering matrices, (2.1).

The exact solution of the model is now obtained following a standard procedure [11, 15, 16]. On imposing periodic boundary conditions, the N_e -particle problem reduces to the simultaneous solution of a set of N_e eigenvalue equations. This eigenvalue problem has been solved by Sutherland [16] for an arbitrary Young tableau by means of a sequence of additional $(N - 1)$ nested Bethe ansätze. Each Bethe ansatz leads to a new eigenvalue problem with the number of spin components reduced by one and gives rise to a set of rapidities. This procedure is repeated until all internal degrees of freedom are eliminated. As a result, N sets of rapidities $\{\xi_\alpha^{(l)}\}$, $l = 0, \dots, N - 1$, are obtained, which are self-consistently determined by the Bethe ansatz equations [11, 12]. The set for $l = 0$ corresponds to the charge rapidities, $\xi_\alpha^{(0)} = p_\alpha = \frac{1}{2} \cot(k_\alpha/2)$, where $\{k_\alpha\}$ are the wavenumbers of the particles, while the other sets are associated with the spin degrees of freedom. All rapidities within a given set have to be different. The rapidities are not independent of each other but coupled by the discrete Bethe ansatz equations [11, 12]

$$\left(\frac{\xi_\alpha^{(0)} - \frac{1}{2}i}{\xi_\alpha^{(0)} + \frac{1}{2}i} \right)^{N_e} = \prod_{\beta=1}^{M_1} \frac{\xi_\alpha^{(0)} - \xi_\beta^{(1)} - \frac{1}{2}i}{\xi_\alpha^{(0)} - \xi_\beta^{(1)} + \frac{1}{2}i} \quad \alpha = 1, \dots, M_0 \tag{2.2a}$$

$$\prod_{\beta=1}^{M_l} \frac{\xi_\alpha^{(l)} - \xi_\beta^{(l)} - i}{\xi_\alpha^{(l)} - \xi_\beta^{(l)} + i} = - \prod_{\beta=1}^{M_{l-1}} \frac{\xi_\alpha^{(l)} - \xi_\beta^{(l-1)} - \frac{1}{2}i}{\xi_\alpha^{(l)} - \xi_\beta^{(l-1)} + \frac{1}{2}i} \prod_{\beta=1}^{M_{l+1}} \frac{\xi_\alpha^{(l)} - \xi_\beta^{(l+1)} - \frac{1}{2}i}{\xi_\alpha^{(l)} - \xi_\beta^{(l+1)} + \frac{1}{2}i}$$

$$l = 1, \dots, N - 1 \quad M_0 \equiv N_e \quad M_N \equiv 0 \quad \alpha = 1, \dots, M_l \tag{2.2b}$$

where N_e is the number of sites in the chain and M_l is the number of rapidities in the set $\{\xi_\alpha^{(l)}\}$. If n_{S-m} denotes the number of electrons with spin component m and $M_{i+1} = M_i - n_i$, then necessarily $N_e \equiv M_0 \geq M_1 \geq M_2 \geq \dots \geq M_{N-1} \geq M_N \equiv 0$. This solution corresponds to the Young tableau $(M_0 - M_1, M_1 - M_2, \dots, M_{N-2} - M_{N-1}, M_{N-1} - M_N)$. Note that the Bethe ansatz eigenfunctions are only a basis of states within this subspace [17]. The energy eigenvalues of the Hamiltonian (1.1) and the magnetization are given by

$$E = -2N_e + 2 \sum_{\alpha=1}^{M_0} \frac{\frac{1}{2}}{(\xi_\alpha^0)^2 + \frac{1}{4}} \tag{2.3a}$$

$$S_z = \frac{1}{2}(N - 1)N_e - \sum_{l=1}^{N-1} M_l. \tag{2.3b}$$

The ground state and the excitations of the system are given by the self-consistent solutions of equations (2.2). The rapidities have, in general, complex values and in the thermodynamic limit (large N_a , N_e and M_l), they can be classified according to [12]

- (i) real charge rapidities, belonging to the set $\{\xi_\alpha^{(0)}\}$, which correspond to unpaired propagating electrons,
- (ii) complex spin and charge rapidities, which correspond to bound states of electrons with different spin components, and
- (iii) strings of complex spin rapidities, which correspond to bound spin states.

In the ground state we only have particles in states corresponding to classes (i) and (ii). The strings of class (iii) represent excited states and will not be considered here. Since only electrons with different spin components are scattered, i.e. experience an effective attractive interaction, we may build spin complexes of up to $(2S + 1)$ electrons. A complex of n electrons ($n \leq 2S + 1$) is characterized by one real $\xi^{(n-1)}$ rapidity and in general complex $\xi^{(l)}$ rapidities, $l < n - 1$, given by

$$\begin{aligned} \xi_p^{(l)} &= \xi^{(n-1)} + (i/2)p \quad l \leq n - 1 \leq 2S \\ p &= -(n - l - 1), -(n - l - 3), \dots, (n - l - 1). \end{aligned} \quad (2.4)$$

These spin and charge strings form classes (i) and (ii). There are then $N = 2S + 1$ sets of real rapidities $\{\xi_\alpha^{(n-1)}\}$ in the ground state. Here α is the running index within a set. All rapidities within a given set have to be different. This property leads to Fermi statistics for rapidities associated with charges and spin waves, although spin waves have an integer spin and are actually hard-core bosons.

The above rapidities are inserted into equations (2.2) and the resulting coupled equations for the real $\{\xi_\alpha^{(l)}\}$ are logarithmized. This generates a set of integer quantum numbers for each set of rapidities. In the thermodynamic limit we define the usual distribution functions $\rho^{(l)}(\xi)$ for the real rapidities $\xi_\alpha^{(l)}$ and similarly the 'hole' distribution functions $\rho_h^{(l)}(\xi)$. 'Particle' and 'hole' densities are not independent in view of the Fermi statistics of the rapidities, but coupled by sets of linear integral equations. Fourier-transforming the equations, we have

$$\begin{aligned} \hat{\rho}_h^{(l)}(\omega) + \hat{\rho}^{(l)}(\omega) + \sum_{q=0}^{2S} \hat{\rho}^{(q)}(\omega) \exp \left[-\frac{|\omega|}{2}(l + q - p_{l,q}) \right] \frac{\sinh \left[\frac{1}{2}\omega(p_{l,q} + 1) \right]}{\sinh(\frac{1}{2}\omega)} \\ = \exp(- (l + 1)|\omega|/2). \end{aligned} \quad (2.5)$$

Here the caret denotes a Fourier transform, $p_{l,q} = \min(l, q) - \delta_{l,q}$ and $l = 0, \dots, 2S$. The energy of the system and the number of particles (per site) for each spin component, n_l , are given by

$$E = -2N_e + 2N_a \sum_{m=0}^{2S} \int d\xi \rho^{(m)}(\xi) \frac{\frac{1}{2}(m+1)}{\xi^2 + \frac{1}{4}(m+1)^2} \quad (2.6)$$

$$n_l = \sum_{q=2S-l}^{2S} \int d\xi \rho^{(q)}(\xi) \quad (2.7)$$

and the total number of electrons becomes

$$n = N_e/N_a = \sum_{l=0}^{2S} n_l = \sum_{l=0}^{2S} (l+1) \int d\xi \rho^{(l)}(\xi). \quad (2.8)$$

These equations differ only in their driving terms (independent terms) and the expression of the energy from the corresponding ones for the N -component one-dimensional fermion gas with attractive δ -function interaction [18] and the degenerate Anderson impurity in the $U \rightarrow \infty$ limit [19].

In the ground state the rapidities are densely distributed. Minimization of the energy yields that the densities are symmetric functions of ξ about $\xi = 0$ with $\rho^{(l)}(\xi) \equiv 0$ for $|\xi| < B_l$ for $l = 0, \dots, 2S$. The $\rho_b^{(l)}(\xi)$ are the complementary functions, which vanish for $|\xi| > B_l$. The integration limits B_l are determined by the number of particles with each spin component, equation (2.7). The $l = 2S$ rapidities correspond to zero-spin clusters of N electrons, each electron having a different spin component. In the absence of external potentials (magnetic or crystalline fields) the Fermi sea consists only of these clusters (assuming that the number of electrons is a multiple of N), represented by the strings of length $2S$. In this limit all B_l are infinite, except B_{2S} which we denote by Q . Q is determined by the band filling; $Q = 0$ corresponds to the full band, while if $Q = \infty$ the band is empty.

To study the response of the ground state of the system to a small level splitting, as well as to obtain the spectrum of elemental excitations, it is convenient to rewrite (2.5) in the following form

$$\begin{aligned} \rho^{(2S)}(\xi) + \rho_b^{(2S)}(\xi) + \int_{-Q}^Q d\xi' \rho_b^{(2S)}(\xi') G(\xi' - \xi) \\ = - \sum_{m=0}^{2S-1} \left[\int_{B_m}^{\infty} + \int_{-\infty}^{-B_m} \right] d\xi' \rho^{(m)}(\xi') F_m(\xi' - \xi) + D(\xi) \end{aligned} \quad (2.9)$$

$$\begin{aligned} \rho_b^{(m)}(\xi) + \sum_{q=0}^{2S-1} \left[\int_{B_q}^{\infty} + \int_{-\infty}^{-B_q} \right] d\xi' \rho^{(q)}(\xi') \\ \times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp \left[i(\xi - \xi')\omega + \frac{|\omega|}{2} \right] \hat{G}_{m+1, q+1}(\omega) \\ = \int_{-Q}^Q d\xi' \rho_b^{(2S)}(\xi') F_m(\xi - \xi') + F_{2S-m}(\xi) \end{aligned} \quad (2.10)$$

where

$$G(\xi) = \frac{1}{N\pi} \operatorname{Re} \left\{ \psi \left(\frac{1}{N} + i\frac{\xi}{N} \right) - \psi \left(1 + i\frac{\xi}{N} \right) \right\} \quad (2.11)$$

$$D(\xi) = \frac{1}{N\pi} \operatorname{Re} \left\{ \psi \left(\frac{1}{2} + \frac{1}{N} + i\frac{\xi}{N} \right) - \psi \left(\frac{1}{2} + i\frac{\xi}{N} \right) \right\} \quad (2.12)$$

$$F_m(\xi) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\xi\omega} \frac{\sinh(\frac{1}{2}\omega(m+1))}{\sinh(\frac{1}{2}\omega N)} = \frac{1}{N} \frac{\sin[\pi(m+1)/N]}{\cosh[2\pi\xi/N] + \cos[\pi(m+1)/N]} \quad (2.13)$$

and

$$\hat{G}_{l,q}(\omega) = \sinh\left(\frac{1}{2}\omega \min(q, l)\right) \sinh\left(\frac{1}{2}\omega(N - \max(q, l))\right) / \sinh\left(\frac{1}{2}\omega N\right) \sinh\left(\frac{1}{2}\omega\right). \quad (2.14)$$

Here Re denotes real part and ψ is the digamma function. The energy can be expressed in terms of the hole states for strings of length $2S$

$$E = -2N\epsilon + \frac{2N_a}{N} \left\{ \psi\left(1 + \frac{1}{N}\right) - \psi(1) \right\} - 2\pi N_a \int_{-Q}^Q d\xi \rho_h^{(2S)}(\xi) D(\xi) \\ + 2\pi N_a \sum_{m=0}^{2S-1} \int d\xi \rho^{(m)}(\xi) F_{2S-m}(\xi). \quad (2.15)$$

For $N = 2$ these equations reduce to the Bethe *ansatz* equations derived previously in [6, 8] for the traditional supersymmetric t - J model. For arbitrary N , the kernel of the integral equations (2.10), i.e. $\hat{G}_{l,q}(\omega)$, has the characteristic form dictated by the $SU(N)$ invariance. This kernel appeared previously in related problems: (i) the $SU(N)$ -invariant Heisenberg chain [5, 20], (ii) the N -colour Fermi gas in one-dimension interacting via a δ -function potential [18, 21], (iii) the Coqblin-Schrieffer model [22, 23], and the degenerate Anderson model with excluded multiple occupancy of the f level [19].

An alternative formulation of the Bethe *ansatz* equations for the ground state is in terms of the thermodynamic energies for each set of rapidities. The thermodynamic energies are defined as

$$\rho_h^{(l)} / \rho^{(l)} = \exp(\epsilon_l / T) \quad (2.16)$$

in the limit $T \rightarrow 0$. For a small Zeeman splitting they satisfy the following set of integral equations [12]:

$$\epsilon_{2S}(\xi) + \int_{-Q}^Q d\xi' \epsilon_{2S}(\xi') G(\xi' - \xi) \\ = -2 - \mu + 2\pi D(\xi) - \sum_{q=0}^{2S-1} \left[\int_{B_q}^{\infty} + \int_{-\infty}^{-B_q} \right] d\xi' \epsilon_q(\xi') F_q(\xi' - \xi) \quad (2.17)$$

$$\epsilon_m(\xi) + \sum_{q=0}^{2S-1} \left[\int_{B_q}^{\infty} + \int_{-\infty}^{-B_q} \right] d\xi' \epsilon_q(\xi') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp\left[i(\xi - \xi')\omega + \frac{|\omega|}{2}\right] \hat{G}_{m+1,q+1}(\omega) \\ = \int_{-Q}^Q d\xi' \epsilon_{2S}(\xi') F_m(\xi - \xi') + 2\pi F_{2S-m}(\xi) - (m+1)(2S-m) \frac{H}{2} \quad (2.18)$$

where $m = 0, \dots, 2S-1$ and μ is the chemical potential. The $\epsilon_m(\xi)$ are symmetric functions of ξ , which decrease monotonically with increasing $|\xi|$. The functions $\epsilon_m(\xi)$ are typically positive for $\xi = 0$ and negative for large ξ . From their definition, equation (2.16), the functions $\epsilon_m(\xi)$ have zeroes at $\xi = \pm B_m$. This defines the relationship of the chemical potential and the magnetic field with the integration limits B_m . Note as well that the integral equations determining the $\rho^{(l)}$ and the ϵ_l

are the same except for the driving terms. The density functions for particles and holes can be derived from the thermodynamic potentials if we modify the driving terms in equations (2.17) and (2.18) by replacing

$$2\pi D(\xi) \rightarrow 2\pi x D(\xi) \quad 2\pi F_{2S-m}(\xi) \rightarrow 2\pi x F_{2S-m}(\xi).$$

Differentiating (2.17) and (2.18) with respect to x , we obtain by comparison with equations (2.9) and (2.10) that

$$\begin{aligned} \rho^{(m)}(\xi) &= (1/2\pi)\partial\epsilon_m/\partial x & \text{if } |\xi| > B_m \\ \rho_h^{(m)}(\xi) &= (1/2\pi)\partial\epsilon_m/\partial x & \text{if } |\xi| < B_m \end{aligned} \tag{2.19}$$

with $B_{2S} = Q$.

In the following three sections we discuss properties of the ground state and elemental excitations of the degenerate supersymmetric t - J model which we obtain by numerically solving the Bethe *ansatz* equations stated above.

3. Ground state properties

In the absence of magnetic and crystalline fields, $B_m = \infty$ for $m = 0, \dots, 2S - 1$. Hence, equations (2.9) and (2.10) reduce to a single integral equation for $\rho^{(2S)}(\xi)$ (we now suppress the superscript (2S) in this section). The band filling is controlled by the parameter Q , and decreases with increasing Q ($Q = 0$ corresponds to the full band, $N_e = N_a$, while for $Q = \infty$ the band is empty). Discretizing the integral (using about 100 points), the integral equation is then straightforwardly solved numerically by iteration. Note that both $\rho(\xi)$ and $\rho_h(\xi)$ are symmetric and non-negative (since they represent densities of states) functions of ξ .

We first consider the empty and full band limits ($n = N_e/N_a = 0$ and 1, respectively), which can be treated analytically.

(i) Empty band. For $n = 0$ we have $Q = \infty$ and the solution of the integral equation for $\rho_h(\xi)$ can straightforwardly be obtained by Fourier transformation

$$\rho_h(\xi) = (1/\pi)(N/2)/[\xi^2 + (N/2)^2]. \tag{3.1}$$

The energy of the system is of course zero in this limit and the chemical potential $\mu = -2$.

(ii) Nearly empty band. If n is small but non-zero (low electron density), then Q is large but finite, and the integral equation can be solved iteratively by reducing it to a sequence of Wiener-Hopf integral equations. The chemical potential μ is then slightly above -2 . After a lengthy calculation we obtain [6, 11, 12]

$$Q = (2 + \mu)^{-1/2} \left\{ 1 + (S/4\pi)(2 + \mu)^{1/2} \ln(2 + \mu) + \dots \right\}$$

and to leading order the number of electrons and the energy are given by

$$n = N_e/N_a = (N/\pi)(2 + \mu)^{1/2} \tag{3.2}$$

$$E/N_a = -2n + \pi^2 n^3/3N^2 \tag{3.3}$$

as expected from the van Hove singularity of a (one-dimensional) free-electron density of states.

(iii) Full band. The situation of a full band can also be treated analytically. No holes corresponds to $Q = 0$, so that $\rho_h \equiv 0$ and

$$\rho(\xi) = (1/\pi N) \operatorname{Re} \left\{ \psi \left(\frac{1}{2} + 1/N + i\xi/N \right) - \psi \left(\frac{1}{2} + i\xi/N \right) \right\}. \quad (3.4)$$

The following relations are easy to verify

$$\begin{aligned} \frac{N_e}{N_a} &= N \int_{-\infty}^{\infty} d\xi \rho(\xi) = 1 & \frac{E}{N_a} &= \frac{2}{N} \left\{ \psi \left(1 + \frac{1}{N} \right) - \psi(1) \right\} - 2 \\ \mu &= 2/N \left\{ \psi \left(\frac{1}{2} + 1/N \right) - \psi \left(\frac{1}{2} \right) \right\} - 2. \end{aligned} \quad (3.5)$$

(iv) Almost full band. If Q is small but finite the system has a low density of holes. In this case the integral equation (2.9) can be solved iteratively [6]. We obtain for the number of electrons and the energy

$$\begin{aligned} n &= N_e/N_a = 1 - (2Q/\pi N) \left\{ \psi \left(\frac{1}{2} + 1/N \right) - \psi \left(\frac{1}{2} \right) \right\} \\ E/N_a &= (2/N) \left[\psi \left(1 + \frac{1}{N} \right) - \psi(1) \right] - 2 \\ &\quad - \left\{ (2/N) \left[\psi \left(\frac{1}{2} + 1/N \right) - \psi \left(\frac{1}{2} \right) \right] - 2 \right\} [1 - n]. \end{aligned} \quad (3.6)$$

The general solution of the integral equation (2.9) for $\rho_h(\xi)$ can only be obtained numerically. The ground state energy is then calculated via equation (2.15). The energy density E/N_a in units of t and as a function of n is shown in figure 1 for several N . These curves interpolate between the empty and full band results derived above analytically.

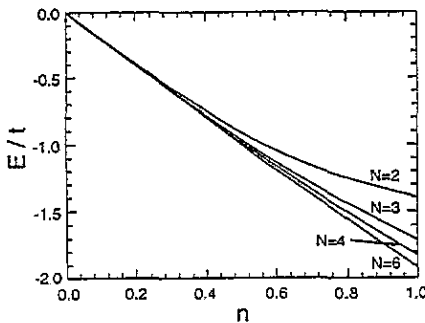


Figure 1. Ground state energy (in units of the hopping matrix element t) in the absence of external fields as a function of the electron density n for $N = 2, 3, 4$ and 6 .

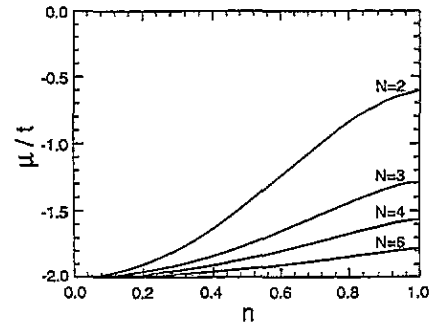


Figure 2. Chemical potential μ (in units of the hopping matrix element t) as a function of the electron density n for $N = 2, 3, 4$ and 6 .

The chemical potential is the energy required to add or remove one electron from the system, i.e. $\mu = \partial E / \partial N_e$. A change in N_e implies a modified integration limit Q and hence a change in $\rho_h^{(2S)}$. Hence, we have [8, 13, 24]

$$\mu = [\partial(E/N_a) / \partial Q] / [\partial n / \partial Q]. \quad (3.7)$$

Denoting $\rho'(\xi) = d\rho(\xi)/dQ$, we have that ρ' obeys the following integral equation:

$$\rho'(\xi) + \rho'_b(\xi) + \int_{-Q}^Q d\xi' \rho'_h(\xi') G(\xi' - \xi) = -[G(\xi - Q) + G(\xi + Q)]\rho(Q). \quad (3.8)$$

Note that $\rho'(\xi)$ is an even function of ξ . The numerical solution of this integral equation is also straightforward using the method described above. One finally obtains that the chemical potential is given by

$$\mu = -2 + 2\pi \left[2D(Q)\rho_h(Q) + \int_{-Q}^Q d\xi \rho'_h(\xi)D(\xi) \right] / \left[2\rho_h(Q) + \int_{-Q}^Q d\xi \rho'_h(\xi) \right]. \quad (3.9)$$

μ as a function of n for several N is displayed in figure 2. These curves interpolate between the empty and full band limits derived above.

Next we discuss the static susceptibility at $T = 0$ for arbitrary band filling in the small Zeeman splitting limit. In a small magnetic field all B_m ($m = 0, \dots, 2S - 1$) are finite but large. The difference in population of Zeeman levels is proportional to H , while the feedback of the field on $\rho^{(2S)}$ and ϵ_{2S} is proportional to H^2 and can be neglected. In other words, we use the zero-field result for $\rho^{(2S)}$ and ϵ_{2S} . The numerical solution of equation (2.17) for ϵ_{2S} in zero field follows in complete analogy with the procedure discussed above for the solution of (2.9) for $\rho^{(2S)}$. For a given Q we determine μ so that $\epsilon_{2S}(\pm Q) = 0$. This is an alternative way to obtain the chemical potential, which agrees with the one discussed above.

Consider now equations (2.18) with the ϵ_{2S} -term (evaluated in zero field) as part of the driving term. For sufficiently small magnetic fields, $B_m \gg Q$ so that the right-hand side of (2.18) can be approximated by

$$\frac{2\pi \sin[\pi(m+1)/N]}{N \cosh[2\pi\xi N]} \left[1 + \frac{1}{2\pi} \int_{-Q}^Q d\xi' \exp\left(\frac{2\pi\xi'}{N}\right) \epsilon_{2S}(\xi') \right] - (m+1)(2S-m) \frac{H}{2}. \quad (3.10)$$

The driving terms of the corresponding set of integral equations obeyed by the density functions $\rho^{(m)}$, equation (2.10), are obtained by dropping the field-dependent driving term in (3.10) and using the relations (2.19). Using the definition of the magnetization it is possible to show that the zero-field and low-field magnetic susceptibilities of the system are given by [6, 20, 21]

$$\chi(0) = (1/4\pi^2)[N(N^2 - 1)/6] \frac{1 + \int_{-Q}^Q d\xi' \exp(2\pi\xi'/N) \rho_h^{(2S)}(\xi')}{1 + (1/2\pi) \int_{-Q}^Q d\xi' \exp(2\pi\xi'/N) \epsilon_{2S}(\xi')} \quad (3.11a)$$

$$\chi(H)/\chi(0) = [1 + 1/(N|\ln H|) + 1/(N|\ln H|)^2 \ln[1/(N|\ln H|)] + \dots]. \quad (3.11b)$$

The limit $n = 1$, i.e. $Q = 0$, corresponds to the $SU(N)$ -invariant Heisenberg chain. In this limit the zero-field susceptibility is given by

$$\chi_{\text{Heis}}(0) = (1/4\pi^2)N(N^2 - 1)/6. \quad (3.12)$$

(Note that the definition of the exchange coupling J here differs by a factor of 2 with respect of that of [20].) The N -dependence of this expression is determined by the $T = 0$ Curie constant of the Heisenberg chain. In figure 3 we show the dependence of $\chi(H = 0)/\chi_{\text{Heis}}$ on the band filling. Since these curves are almost independent of N , only the ones corresponding to $N = 2$ and $N = 6$ are presented. The susceptibility diverges as $n \rightarrow 0$ proportional to $(2 + \mu)^{-1/2}$, as a consequence of the one-dimensional van Hove singularity. In a small but finite field we obtain logarithmic singularities caused by the interference between the two Fermi points (scattering across the Fermi surface) [20, 21].

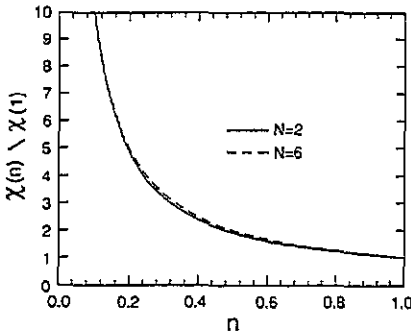


Figure 3. Zero-field static susceptibility, $\chi(n)$, normalized to its value for $n = 1$ (Heisenberg chain) as a function of electron density n for $N = 2$ and 6. Note that the curve is almost universal with N . The susceptibility diverges as $n \rightarrow 0$ as a consequence of the one-dimensional van Hove singularity.

4. Elemental charge excitations (holons)

In this section we study the particle and hole excitation spectrum of charges. Charge excitations are obtained by adding or removing a rapidity from the set $\{\xi_\alpha^{(2S)}\}$. This set corresponds to the spin-zero clusters of N electrons, represented by strings of length N . All other sets of rapidities involve spin excitations. Note that these spin and charge excitations are decoupled; since the integral equations are linear, the superposition principle applies and excitations are additive and independent. Their behaviour is soliton-like. The excitation spectrum for $N = 2$ has been derived in [7, 8]. We present here the generalization to arbitrary spin-degeneracy. We follow a similar procedure to that employed for the one-dimensional Hubbard model by Coll [24] ($N = 2$) and Schlottmann [13] (arbitrary N). We explicitly distinguish *hole* states, i.e. removing one rapidity, and *particle* states, i.e. adding one rapidity. We limit ourselves to the case $B_m = \infty$, $m = 0, \dots, 2S - 1$, i.e. the system in the absence of external potentials.

4.1. Hole states

We first consider the simpler situation of a full band, $n = 1$ and $Q = 0$. In the absence of holes we have

$$\rho^{(2S)}(\xi) = D(\xi) \quad \rho_h^{(2S)}(\xi) \equiv 0.$$

To simplify the notation we drop the superscript $(2S)$ in the string density distribution function. If one hole is introduced with rapidity ξ_0 , we have $\rho_h(\xi) = (1/N_a)\delta(\xi - \xi_0)$. Consequently the rapidities are rearranged and their density changes by the amount $(1/N_a)\Delta\rho(\xi)$. From (2.9) we obtain

$$\Delta\rho(\xi) = -\delta(\xi - \xi_0) - G(\xi - \xi_0) \quad (4.1)$$

and the change in energy associated with the removing of the charge is

$$\begin{aligned} \Delta E_{\text{ch}}(\xi_0) &= (2/N) \operatorname{Re} \left\{ \psi\left(\frac{1}{2} + 1/N\right) - \psi\left(\frac{1}{2}\right) \right. \\ &\quad \left. - \psi\left(\frac{1}{2} + 1/N + i\xi_0/N\right) + \psi\left(\frac{1}{2} + i\xi_0/N\right) \right\}. \end{aligned} \quad (4.2)$$

The excitation energy vanishes for $\xi_0 = 0$ (Fermi level) and increases monotonically with $|\xi_0|$ reaching its maximum of

$$(2/N) \left\{ \psi\left(\frac{1}{2} + 1/N\right) - \psi\left(\frac{1}{2}\right) \right\}$$

as $|\xi_0|$ tends to infinity.

We now consider hole states in a partially filled band. Only rapidities with $|\xi_0| > Q$ can be removed. Removing a charge rapidity introduces an additional driving term, so that the solution of the integral equation is of the form $\rho_h(\xi) + \Delta\rho_h(\xi)/N_a$, where $\rho_h(\xi)$ is the ground state distribution and $\Delta\rho_h(\xi)$ accounts for the rearrangement of the rapidities due to the missing one. $\Delta\rho_h(\xi)$ satisfies the following integral equation:

$$\Delta\rho_h(\xi) = - \int_{-Q}^Q d\xi' \Delta\rho_h(\xi') G(\xi' - \xi) - \delta(\xi - \xi_0) - G(\xi - \xi_0). \quad (4.3)$$

The δ -function on the right-hand side of (4.3) represents the missing ξ -value, while the other terms correspond to the 'self-energy', i.e. the redistribution of the rapidities. We now isolate the δ -function term in the equation for $\Delta\rho_h$ by writing

$$\Delta\rho_h(\xi) = -\delta(\xi - \xi_0) + \Delta\tilde{\rho}_h(\xi). \quad (4.4)$$

Here $\Delta\tilde{\rho}_h$ satisfies the following integral equation:

$$\Delta\tilde{\rho}_h(\xi) = - \int_{-Q}^Q d\xi' \Delta\tilde{\rho}_h(\xi') G(\xi' - \xi) - G(\xi - \xi_0) \quad (4.5)$$

which is similar to the one satisfied by $\rho'_h(\xi)$, equation (3.8). The numerical solution of (4.5) is then straightforward. $\Delta\tilde{\rho}_h$ has the following symmetry

$$\Delta\tilde{\rho}_h(\xi, \xi_0) = \Delta\tilde{\rho}_h(-\xi, -\xi_0).$$

The integral $\int_{-Q}^Q d\xi \Delta\tilde{\rho}_h(\xi)$ does not, in general, vanish. Hence, the integral over $\Delta\rho_h(\xi)$ is not equal to -1 as it should be if exactly one charge has been removed. There is an additional contribution which we have neglected so far. By eliminating one charge the integration limit Q has also to be readjusted [13, 24], since the number of particles has changed. This change of Q is small, of the order of $1/N_a$, but of the same order as $\Delta\rho_h(\xi)$, so that it cannot be neglected. Denoting with Q_0 and Q the integration limits before and after removing the charge, we have invoking the conservation of particles

$$(Q - Q_0) \frac{dn}{dQ} = 1 + \int_{-Q_0}^{Q_0} d\xi \Delta\tilde{\rho}_h(\xi). \quad (4.6)$$

We may either use Q or Q_0 on the right-hand side of (4.6), since the difference is of higher order in $1/N_a$. The energy of the excitation is given by

$$\begin{aligned} \Delta E_{ch}(\xi_0) = & 2 \left(1 + \int_{-Q_0}^{Q_0} d\xi \Delta\tilde{\rho}_h(\xi) \right) \\ & - 2\pi \int_{-Q_0}^{Q_0} d\xi \Delta\tilde{\rho}_h(\xi) D(\xi) - 2\pi D(\xi_0) + E(Q) - E(Q_0) \end{aligned} \quad (4.7)$$

where the last two terms account for the change in ground state energy due to the change in Q . Since $Q - Q_0$ is small, we approximate

$$E(Q) - E(Q_0) = (Q - Q_0) \frac{dE}{dQ} = \mu \left(1 + \int_{-Q_0}^{Q_0} d\xi \Delta\tilde{\rho}_h(\xi) \right) \quad (4.8)$$

where μ is the chemical potential calculated before and we used (4.6) to eliminate $(Q - Q_0)$.

4.2. Particle states

Again we first consider the simpler situation of an empty band, $n = 0$ and $Q = \infty$, which can be solved analytically. In the absence of particles all the density functions $\rho^{(m)}$ are identically zero. Adding the charge rapidity ξ_0 yields then $\rho^{(2S)}(\xi) = \delta(\xi - \xi_0)$ and the change of energy is given by (note that for the empty band $\mu = -2$)

$$\Delta E_{\text{ch}}(\xi_0) = 2(N/2)/[\xi_0^2 + (N/2)^2]. \quad (4.9)$$

The excitation energy vanishes for $|\xi_0| \rightarrow \infty$ (Fermi level) and increases monotonically when $|\xi_0|$ is reduced, reaching its maximum of $4/N$ for $\xi_0 = 0$.

We now consider particle states in a partially filled band. Only rapidities with $|\xi_0| < Q$ can be added. Adding a charge rapidity introduces an additional driving term and changes the distribution of rapidities by an amount $\Delta\rho_h(\xi)/N_a$ which satisfies the following integral equation (we again drop the superscript (2S)):

$$\Delta\rho_h(\xi) = - \int_{-Q}^Q d\xi' \Delta\rho_h(\xi') G(\xi' - \xi) + \delta(\xi - \xi_0). \quad (4.10)$$

The δ -function on the right-hand side of (4.10) represents the added particle and the integral is the self-energy effect due to the rearranged rapidities. We again write

$$\Delta\rho_h(\xi) = \delta(\xi - \xi_0) + \Delta\bar{\rho}_h(\xi) \quad (4.11)$$

where $\Delta\bar{\rho}_h$ satisfies the following integral equation:

$$\Delta\bar{\rho}_h(\xi) = - \int_{-Q}^Q d\xi' \Delta\bar{\rho}_h(\xi') G(\xi' - \xi) - G(\xi - \xi_0). \quad (4.12)$$

This integral equation is the same one satisfied for holes, equation (4.5), except that now $|\xi_0| < Q$. Its numerical solution is then straightforward and $\Delta\bar{\rho}_h(\xi, \xi_0) = \Delta\bar{\rho}_h(-\xi, -\xi_0)$.

The energy of the particle is given by

$$\begin{aligned} \Delta E_{\text{ch}}(\xi_0) = & -2 \left(1 + \int_{-Q_0}^{Q_0} d\xi \Delta\bar{\rho}_h(\xi) \right) \\ & + 2\pi \int_{-Q_0}^{Q_0} d\xi \Delta\bar{\rho}_h(\xi) D(\xi) + 2\pi D(\xi_0) + E(Q) - E(Q_0) \end{aligned} \quad (4.13)$$

where the last two terms account for the change of the integration limit Q due to the addition of one particle. From the conservation of the number of particles, we obtain

$$E(Q) - E(Q_0) = -\mu \left(1 + \int_{-Q_0}^{Q_0} d\xi \Delta\bar{\rho}_h(\xi) \right) \quad (4.14)$$

where μ is the chemical potential.

Note that, except for the overall sign, the expressions for the hole and particle excitation energies are the same, although one is valid for $|\xi_0| > Q$ and the other for $|\xi_0| < Q$. Hence, for $\xi_0 = \pm Q$, their excitation energies are equal and necessarily zero. $\xi_0 = \pm Q$ corresponds to particles and holes at the Fermi level.

4.3. Momentum of charge excitations

Although the charge rapidities or the wavenumbers $\{k_\alpha\}$ are frequently called momenta, they do not represent the physical momentum of the particles and holes. The physical momentum is given by the charge quantum number of the particle added or removed and hence is related to the density functions $\rho^{(2S)}(\xi)$ and $\rho_b^{(2S)}(\xi)$ via their definition [8, 13, 24]

$$p = 2\pi \operatorname{sgn}(\xi_0) \int_{|\xi_0|}^{\infty} d\xi \left(\rho^{(2S)}(\xi) + \rho_b^{(2S)}(\xi) \right). \quad (4.15)$$

With this definition, the momentum of the charges is zero if $|\xi_0| \rightarrow \pm\infty$, i.e. at the centre of the Brillouin zone. (Note that from the *ansatz* for the wavefunction in equations (2.2) the rapidities $\xi_\alpha^{(0)} = \frac{1}{2} \cot(k_\alpha/2)$, where k_α is a wavenumber. Hence, the sign of k_α correlates with the one of ξ and while $|k_\alpha|$ increases, $|\xi|$ decreases. There is a change of branch at $\xi = 0$.) The maximum momentum, p_{\max} , is reached when $\xi_0 \rightarrow 0$.)

$$p_{\max} = \pi - \pi n(N-1)/N. \quad (4.16)$$

The excitation spectrum is then symmetric with p . For $\xi_0 = \pm Q$, equation (4.15) yields $p_F = \pm\pi n/N$, where n is the band filling. Recalling that for $|\xi_0| > Q$ the excitations are holes and for $|\xi_0| < Q$ they are particles, we have that p_F corresponds to the Fermi momentum. It is also clear that the crystal momentum is only defined modulo 2π .

Analytic results can be obtained in the limiting cases of an empty and full band. For the empty band the momentum of the added particle is

$$p = 2 \operatorname{sgn}(\xi_0) [(\pi/2) - \arctan(2\xi_0/N)] \quad (4.17a)$$

while for the full band the momentum of the created hole is given by

$$p = \frac{\pi}{N} + i \ln \left[\frac{\Gamma(\frac{1}{2} + 1/N + i\xi_0/N) \Gamma(\frac{1}{2} - i\xi_0/N)}{\Gamma(\frac{1}{2} + 1/N - i\xi_0/N) \Gamma(\frac{1}{2} + i\xi_0/N)} \right]. \quad (4.17b)$$

Here Γ denotes the Γ function.

4.4. Fermi velocity

The Fermi velocity of the charge excitations is defined as the absolute value of $d\Delta E_{\text{ch}}/dp$ at the Fermi level and is obtained as

$$v_F = \left[\frac{d\Delta E_{\text{ch}}(\xi_0)}{d\xi_0} \right]_Q \bigg/ \left[\frac{dp}{d\xi_0} \right]_Q. \quad (4.18)$$

The denominator is straightforwardly obtained from equation (4.15),

$$[dp/d\xi_0]_Q = 2\pi \rho_b^{(2S)}(Q). \quad (4.19)$$

In order to get the numerator, we differentiate either equation (4.5) or equation (4.12) with respect to ξ_0 . Denoting

$$\Psi(\xi) = (d\Delta \bar{\rho}_h(\xi)/d\xi_0)_{\xi_0=Q}$$

we have that Ψ satisfies the following integral equation

$$\Psi(\xi) = - \int_{-Q}^Q d\xi' \Psi(\xi') G(\xi' - \xi) - \left(\frac{dG(\xi_0 - \xi)}{d\xi_0} \right)_{\xi_0=Q} \quad (4.20)$$

and

$$\begin{aligned} \left[\frac{d\Delta E_{\text{ch}}(\xi_0)}{d\xi_0} \right]_{\xi_0=Q} &= -(2 + \mu) \left(1 + \int_{-Q}^Q d\xi \Psi(\xi) \right) \\ &+ 2\pi \int_{-Q}^Q d\xi \Psi(\xi) D(\xi) + 2\pi \left(\frac{dD(\xi_0)}{d\xi_0} \right)_{\xi_0=Q}. \end{aligned} \quad (4.21)$$

The derivatives of G and D with respect to ξ_0 can be expressed in terms of the imaginary part of trigamma functions.

It is straightforward to see that the Fermi velocity vanishes in both the empty and full band limits.

4.5. Results

The results are obtained by numerically solving the above integral equations. The charge excitation spectrum (in units of the hopping matrix element t) for various band fillings is shown in figures 4(a)–4(d) for $N = 2, 3, 4$ and 6 , respectively. As discussed before, $\Delta E_{\text{ch}} = 0$ for $\xi_0 = Q$, i.e. at the Fermi level $p_F = \pi n/N$. Close to the Fermi level, the excitation energy is proportional to $(p - p_F)$, the proportionality constant being the Fermi velocity. The Fermi velocity is non-zero if $n \neq 0$ or $n \neq 1$, the latter is implied by the fact that the system is an insulator by construction. The Fermi velocity (in units of t) for $N = 2, 3, 4$ and 6 as a function of band filling is plotted in figure 5. The height of the maximum of v_F decreases with N .

The particle–hole continuum for the charge excitations is obtained by adding the energies and momenta of a particle excitation with a hole excitation. The momentum of the combined excitation should be considered modulo 2π . For $N = 2$ there are excitations for every momentum $p = p_1 + p_2$. This is no longer the case if $N > 2$, where we have intervals in momentum space for which there are no charge particle–hole excitations.

5. Spinwave excitations

In this section we study elemental spin excitations for the degenerate situation (spin singlet state), i.e. in the absence of external fields, so that $B_m = \infty$ for $m = 0, \dots, 2S - 1$. In other words, the sets of spin rapidities (strings of length less than $2S$) are empty, so that it is only possible to add one rapidity (particle excitations), but not to remove a rapidity (hole excitation). The ground state consists of an equal number of electrons with each spin component. Adding one $\xi^{(m)}$ rapidity ($m < 2S$) corresponds to a spinwave excitation with the simultaneous addition of $(m + 1)$ electrons (see equation (2.8)) [6–8, 13, 24].

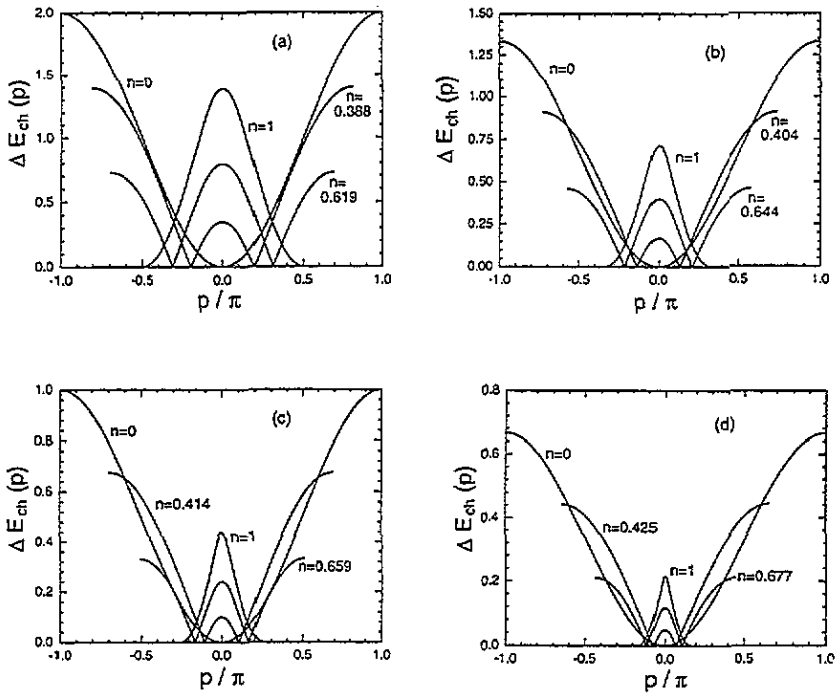


Figure 4. Charge excitation energy (in units of the hopping matrix element t) as a function of crystal momentum over π for (a) $N = 2$, (b) $N = 3$, (c) $N = 4$ and (d) $N = 6$. In each case the dispersion for the empty and full bands, as well as two intermediate electron densities are shown. ΔE_{ch} vanishes at the Fermi surface. $|p| < \pi n$ corresponds to hole states, while $|p| > \pi n$ corresponds to particle states. The range of the spectrum is confined to $|p|/\pi \leq 1 - n + n/N$.

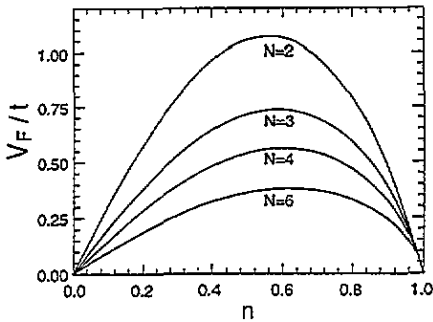


Figure 5. Fermi velocity in units of the hopping matrix element t as a function of band filling n for $N = 2, 3, 4$ and 6 . The system is metallic except if $n = 0$ and $n = 1$.

5.1. Elemental spinwaves

Consider $B_m = \infty$ for all m . Adding $\xi_0^{(m)}$ yields

$$\rho^{(m)}(\xi) = \delta(\xi - \xi_0^{(m)}) \tag{5.1}$$

which gives rise to a rearrangement of the charge rapidities via equation (2.9). The change in the density of charge rapidities is given by

$$\Delta \rho_b^{(2S)}(\xi) = - \int_{-Q}^Q d\xi' \Delta \rho_h^{(2S)}(\xi') G(\xi' - \xi) - F_m(\xi - \xi_0^{(m)}) \tag{5.2}$$

where $F_m(\xi)$ is defined by (2.13). Again, this linear integral equation has to be solved numerically by discretizing the interval. Since, in general,

$$\int_{-Q}^Q d\xi \Delta \rho_h^{(2S)}(\xi) \neq 0 \quad (5.3)$$

the integration limit Q has to be adjusted so that the number of electrons is conserved [8, 13, 24]. The energy of the spinwave excitation is then given by

$$\begin{aligned} \Delta E_{sw}^{(m)}(\xi_0^{(m)}) &= 2 \int_{-Q}^Q d\xi \Delta \rho_h^{(2S)}(\xi) - 2\pi \int_{-Q}^Q d\xi \Delta \rho_h^{(2S)}(\xi) D(\xi) \\ &+ 2\pi F_{2S-m}(\xi_0^{(m)}) + E(Q) - E(Q_0) \end{aligned} \quad (5.4)$$

where the meaning of $E(Q)$ and $E(Q_0)$ is the same as in section 4. The conservation of electrons requires that

$$E(Q) - E(Q_0) = \mu \int_{-Q}^Q d\xi \Delta \rho_h^{(2S)}(\xi) \quad (5.5)$$

where μ is the chemical potential. From the definitions of $G(\xi)$ and $F_m(\xi)$, equations (2.11) and (2.13), we obtain that $\Delta \rho_h^{(2S)}$ has the following symmetry property:

$$\Delta \rho_h^{(2S)}(\xi, \xi_0^{(m)}) = \Delta \rho_h^{(2S)}(-\xi, -\xi_0^{(m)})$$

so that the energy is an even function of $\xi_0^{(m)}$. Since there are $2S$ internal degrees of freedom, there are $2S$ elemental spinwave branches, one corresponding to each set of rapidities, $m = 0, \dots, 2S - 1$. These excitations are soliton-like, i.e. the energy of a finite number of excitations is the sum of the individual excitation energies.

Again, analytic results can be obtained in the two limiting cases $n = 0$ and $n = 1$. For the empty band ($\mu = -2$) the excitation energies are

$$\Delta E_{sw}^{(m)}(\xi_0^{(m)}) = 2(m/2) / \{(\xi_0^{(m)})^2 + [(m+1)/2]^2\} \quad (5.6)$$

while for the full band we have

$$\Delta E_{sw}^{(m)}(\xi_0^{(m)}) = \frac{2\pi}{N} \frac{\sin[\pi(m+1)/N]}{\cosh[2\pi\xi_0^{(m)}/N] - \cos[\pi(m+1)/N]} \quad (5.7)$$

In all cases the energy is a maximum if $\xi_0^{(m)} = 0$ and tends to zero if $|\xi_0^{(m)}|$ tends to infinity.

5.2. Momentum of spinwave excitations

In order to obtain the dispersion relations for the spinwaves, we must find the relation of the momentum to the parameter $\xi_0^{(m)}$. The momentum of the spinwave is given by the quantum number corresponding to the added rapidity $\xi_0^{(m)}$. From the definition of $\rho_h^{(m)}(\xi)$, we have that

$$p_m = 2\pi \int_{-\infty}^{\xi_0^{(m)}} d\xi \rho_h^{(m)}(\xi). \quad (5.8)$$

Using (2.10) with $B_m = \infty$ for $m = 0, \dots, 2S - 1$ we obtain

$$p_m(\xi_0^{(m)}) = 2 \arctan \left\{ \tan \left(\frac{\pi}{2} \frac{2S - m}{N} \right) \tanh \left[\frac{\pi}{N} \xi_0^{(m)} \right] \right\} + \frac{\pi}{N} (2S - m) \\ + 2 \int_{-Q}^Q d\xi \rho_h^{(2S)}(\xi) \arctan \left\{ \tan \left(\frac{\pi}{2} \frac{m + 1}{N} \right) \tanh \left[\frac{\pi}{N} (\xi_0^{(m)} - \xi) \right] \right\} \\ + \pi(m + 1)(1 - n)/N \quad (5.9)$$

by straightforward integration. From the definition, equation (5.8), it is clear that $p_m \rightarrow 0$ for $\xi_0^{(m)} \rightarrow -\infty$. For $\xi_0^{(m)} = 0$ we obtain $p_m = \pi - \pi n(m + 1)/N$, which corresponds to the maximum of the spinwave dispersion. As $\xi_0^{(m)} \rightarrow +\infty$, $\Delta E_{sw}^{(m)} \rightarrow 0$ and $p_m = 2\pi - 2\pi n(m + 1)/N$, which is the maximum momentum of the branch.

Analytic results are again available in the limiting cases of an empty and full band. If the band is empty ($Q = \infty$)

$$p_m(\xi_0^{(m)}) = 2 \arctan \left(2\xi_0^{(m)} / (m + 1) \right) + \pi \quad (5.10a)$$

while for the full band we have

$$p_m(\xi_0^{(m)}) = 2 \arctan \left\{ \tan \left(\frac{\pi}{2} \frac{2S - m}{N} \right) \tanh \left[\frac{\pi}{N} \xi_0^{(m)} \right] \right\} + \frac{\pi}{N} (2S - m). \quad (5.10b)$$

5.3. Spinwave velocity

We now analyse the long-wavelength limit of the spinwave excitation spectrum. As $\xi_0^{(m)} \rightarrow -\infty$, the driving term of equation (5.2) can be expanded in powers of $\exp(2\pi\xi_0^{(m)}/N)$. Defining φ so that

$$\varphi(\xi) = \Delta \rho_h^{(2S)}(\xi) \exp(-2\pi\xi_0^{(m)}/N) / [2 \sin(\pi(m + 1)/N)] \quad (5.11)$$

we obtain the following differential equation for $\varphi(\xi)$:

$$\varphi(\xi) = - \int_{-Q}^Q d\xi' \varphi(\xi') G(\xi' - \xi) - \frac{1}{N} \exp(-2\pi\xi/N) \quad (5.12)$$

which is now independent of m and $\xi_0^{(m)}$. The long-wavelength excitation energy is given by

$$\Delta E_{sw}^{(m)}(\xi_0^{(m)} \rightarrow -\infty) = 2 \sin \left(\pi \frac{m + 1}{N} \right) \exp(2\pi\xi_0^{(m)}/N) \\ \times \left\{ (2 + \mu) \int_{-Q}^Q d\xi \varphi(\xi) - 2\pi \int_{-Q}^Q d\xi \varphi(\xi) D(\xi) + \frac{2\pi}{N} \right\}. \quad (5.13)$$

In the $\xi_0^{(m)} \rightarrow -\infty$ limit, equation (5.9) can be approximated by

$$p_m(\xi_0^{(m)} \rightarrow -\infty) = 2 \sin(\pi(m + 1)/N) \\ \times \exp(2\pi\xi_0^{(m)}/N) \left[1 + \int_{-Q}^Q d\xi \rho_h^{(2S)}(\xi) \exp(-2\pi\xi/N) \right] \quad (5.14)$$

so that $\Delta E_{sw}^{(m)}$ and p_m are proportional and the spinwave velocity is obtained as the ratio of (5.13) and (5.14), $v_{sw} = \Delta E_{sw}^{(m)}/p_m$

$$v_{sw} = \frac{(2 + \mu) \int_{-Q}^Q d\xi \varphi(\xi) - 2\pi \int_{-Q}^Q d\xi \varphi(\xi) D(\xi) + 2\pi/N}{1 + \int_{-Q}^Q d\xi \rho_h^{(2S)}(\xi) \exp(-2\pi\xi/N)}. \quad (5.15)$$

Note that v_{sw} is independent of m ; hence, all spinwave branches have the same continuum limit, i.e. the same spinwave velocity. For $N = 2$ the spinwave velocity agrees with the result in [8].

It is straightforward to see that for the full band ($Q = 0$, i.e. the Heisenberg limit; note that our definition of J differs by a factor of 2 from the one of [20]) the spinwave velocity is $v_{sw} = 2\pi/N$, while $v_{sw} = 0$ for the empty band.

The spinwave velocity is inversely proportional to the spin-susceptibility derived in section 3 (see [20]). Their product is equal to

$$\chi_s v_{sw} = (N^2 - 1)/12\pi \quad (5.16)$$

for all band fillings. This result is not unexpected, since the long-wavelength spin excitations determine χ_s .

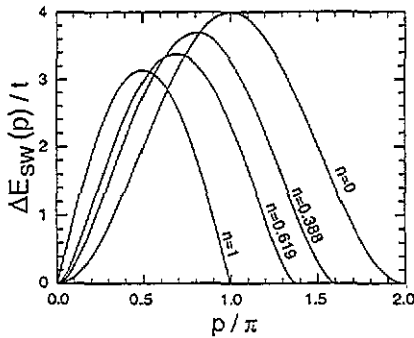


Figure 6. Spinwave excitation energy (in units of the hopping matrix element t) as a function of crystal momentum over π for $N = 2$ and several band fillings. The range of the momentum interval of the excitation depends on the electron density and is related to the Fermi surface of the charges.

5.4. Results

Our numerical results for the spinwave dispersions for $N = 2$ and four band fillings are shown in figure 6. They are in agreement with those presented in [7, 8]. There is only one spinwave branch for $N = 2$. In general there are $N - 1$ spinwave branches. We normalize the excitation energy in units of the hopping matrix element t . The two spinwave branches for $N = 3$ are shown in figures 7(a) and 7(b), respectively, for the empty and the full bands and two intermediate band fillings. As discussed above, $\Delta E_{sw}^{(m)} = 0$ for $p_m = 0$ and $p_m = 2\pi - 2\pi n(m + 1)/N$. The momentum of the excitations is of course only defined modulo 2π . The range of momenta for which there are spinwave excitations is linked to the Fermi surface. The spin excitation spectrum for $N = 4$ and $N = 6$ is shown in figures 8(a)–(d) and figures 9(a)–(d), respectively, for several electron densities. For $n = 1$ the spectrum is identical to the one of the $SU(N)$ -invariant Heisenberg chain.

For small p the excitation energy is proportional to the momentum (except for the empty band), the proportionality constant being the spinwave velocity. The spinwave velocity (in units of t) for $N = 2, 3, 4$ and 6 is plotted as a function of band filling in

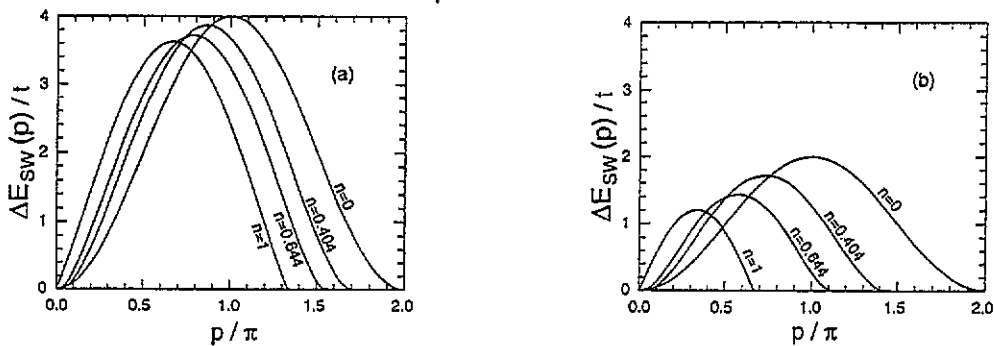


Figure 7. Spinwave excitation energy (in units of the hopping matrix element t) as a function of crystal momentum over π for $N = 3$ and several band fillings. The range of the momentum interval of the excitation depends on the electron density and is related to the Fermi surface of the charges. The two branches $l = 0$ and $l = 1$ are shown in (a) and (b), respectively.

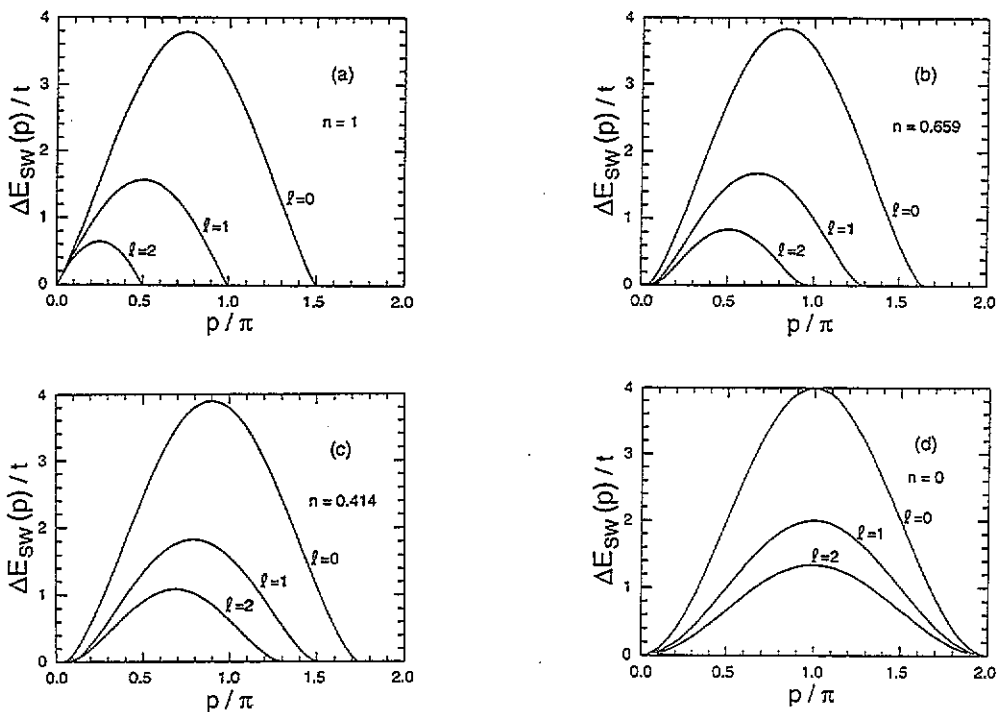


Figure 8. The three branches of the spinwave excitation energy (in units of the hopping matrix element t) for $N = 4$ as a function of crystal momentum over π for (a) $n = 1$, (b) $n = 0.659$, (c) $n = 0.414$ and (d) $n = 0$. The range of the momentum interval of the excitation depends on the electron density and is related to the Fermi surface of the charges.

figure 10. v_{sw} is proportional to the electron density for small n . This proportionality actually holds over quite a large density range. The spinwave velocity decreases with N .

Since all integral equations are linear in the densities and so is the energy, the

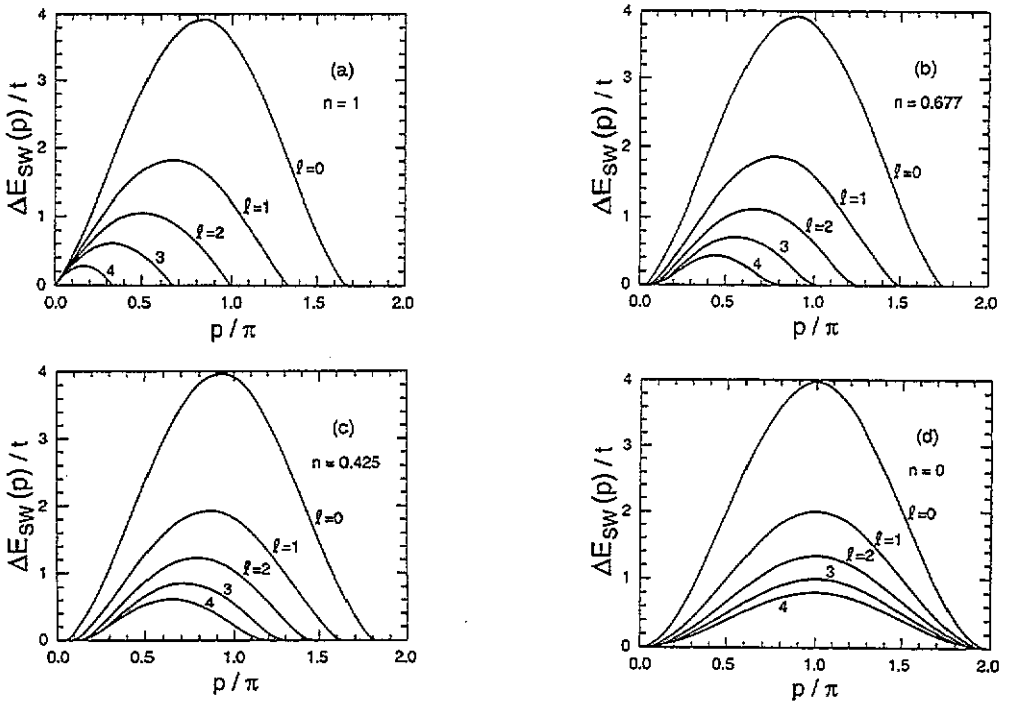


Figure 9. The five branches of the spinwave excitation energy (in units of the hopping matrix element t) for $N = 6$ as a function of crystal momentum over π for (a) $n = 1$, (b) $n = 0.677$, (c) $n = 0.425$ and (d) $n = 0$. The range of the momentum interval of the excitation depends on the electron density and is related to the Fermi surface of the charges.

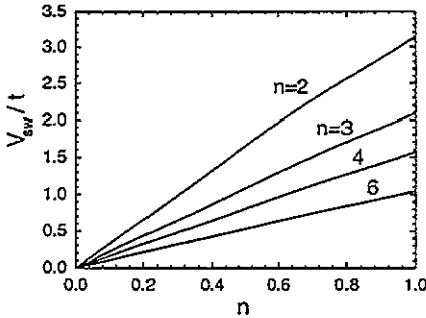


Figure 10. Spinwave velocity v_{sw} in units of the hopping matrix element t as a function of band filling for $N = 2, 3, 4$ and 6 . All branches of the spinwave spectrum have the same spinwave velocity (long wavelength limit).

superposition principle holds and the energy and momenta of excitations are additive. The momentum of the combined excitation should be considered modulo 2π . It is now possible to combine charge with spin excitations, e.g., to replace a charge string of length $2S$ by two spinwave excitations, one represented by a string of length m and one of length $(2S - m)$.

6. Concluding remarks

We considered the N -fold degenerate t - J model, which is completely integrable at

the supersymmetric point $t = J = -V$ [11, 12]. The components may be thought of as arising from combined spin and orbital degrees of freedom. Note that the model can be mapped onto a $(N + 1)$ -component quantum lattice gas as introduced by Sutherland [5]. The additional degree of freedom corresponds to the charges. We extracted the solutions of the Bethe *ansatz* equations relevant to the ground state of the Hamiltonian. They correspond to strings of length m , where $m = 0, \dots, 2S$. These string states represent spin-charge bound states of electrons with different spin components. In the absence of external potentials (magnetic and crystalline fields), and assuming that the total number of electrons is a multiple of N , the ground state is a spin-singlet and all particles are part of string states of length N .

Although we classified the complex spin and charge rapidities (2.4) as bound states of electrons with different spin components, they are not bound states in the real sense, since no actual binding energy is involved (the binding energy is zero). This is in close analogy to the degenerate Anderson model for a magnetic impurity ($U \rightarrow \infty$ limit) [19], where similar string states form a complex representation of a free electron gas.

We have numerically solved the ground state Bethe *ansatz* equations in the absence of external fields for several band fillings and degeneracies N . We obtained the ground state energy, the chemical potential and the zero-field susceptibility. The ground state energy decreases monotonically with the number of electrons, while the μ increases with n . For a full band the charges do not have dynamics and the equations reduce to those of the $SU(N)$ -invariant Heisenberg chain. The zero-field susceptibility is a decreasing function of the electron density. χ is expected to diverge when $N_e/N_a \rightarrow 0$ (or $\mu \rightarrow -2$) as $(2 + \mu)^{-1/2}$ as a consequence of the 1D van Hove singularity. In a finite but small magnetic field the susceptibility shows logarithmic singularities, which are characteristic of one-dimensional systems with $SU(N)$ -symmetry and arise due to the interference of the two spinwave Fermi-surface points [6, 20, 21] (there is only a Fermi surface for the spinwaves if at least one $B_m \neq \infty$). The specific heat at low T is proportional to T . The proportionality constant γ diverges as $(2 + \mu)^{-1/2}$ with $n \rightarrow 0$ ($\mu \rightarrow -2$) due to the one-dimensional van Hove singularity. As a consequence of the logarithmic field singularities the γ coefficient is singular for all band fillings, in the sense that the limits $T \rightarrow 0$ and $H \rightarrow 0$ cannot be interchanged [25]. In particular, if $n = 1$ and $N = 2$ we have [25]

$$\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} \gamma(H, T) = \frac{1}{6} \left(1 + \sqrt{e/\pi} \right) \neq \lim_{T \rightarrow 0} \lim_{H \rightarrow 0} \gamma(H, T) = \frac{1}{3}.$$

(Note that our definition of J differs by a factor of 2 from the one in [25].)

In sections 4 and 5 we derived the charge and spin excitation spectrum of the system in the absence of external potentials. This generalizes the results by Bares and Blatter [7] for $N = 2$ (the traditional t - J model) to arbitrary degeneracy N . The system is metallic, i.e. the Fermi velocity is finite, except for a totally empty or full band. There are intervals in momentum space for which there are no particle-hole charge excitations. There are $N - 1$ spinwave branches with a common long-wavelength spinwave velocity. v_{sw} increases monotonically with the electron density n ; for small n we have that v_{sw} is proportional to n . Again, the momentum interval for which spinwaves are defined is linked to the Fermi surface of the charges. The spinwave velocity is inversely proportional to the zero-field static susceptibility.

Since the expression of the energy and the integral equations governing the excitation spectra are linear in the rapidity density functions, the superposition

principle holds for any finite number of excitations. The excitation energies and their momenta are additive, leading to a continuum of excitations for given momentum. With several branches and many possible combinations of charge and spin excitations, the continuum of excitations becomes very complicated.

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